Midterm test for Kwantumfysica 1 - 2010-2011

Friday 10 December 2010, 15:00 - 16:00

READ THIS FIRST:

- Clearly write your name and study number on each answer sheet that you use.
- On the first answer sheet, write clearly the total number of answer sheets that you turn in.
- Note that this test has 3 questions, it continues on the backside of the paper!
- Start each question (number T1, T2, T3) on a new side of an answer sheet.
- The test is open book within limits. You are allowed to use the book by Liboff, and one A4 sheet with notes, but nothing more than this.
- If you get stuck on some part of a problem for a long time, it may be wise to skip it and try the next part of a problem first. The test is only 60 minutes.

Useful formulas and constants:

Electron mass	m_e	$= 9.1 \cdot 10^{-31}$ kg
Electron charge	$-e$	$= -1.6 \cdot 10^{-19}$ C
Planck's constant	h	$= 6.626 \cdot 10^{-34}$ Js $= 4.136 \cdot 10^{-15}$ eVs
Planck's reduced constant	$\hbar$	$= 1.055 \cdot 10^{-34}$ Js $= 6.582 \cdot 10^{-16}$ eVs

Problem T1

Consider a quantum system with one degree of freedom that is the motion of a point particle along a direction x. This system has a stationary Hamiltonian.

a) Derive for this system the time-independent Schrödinger equation from the time-dependent Schrödinger equation.

b) Write down (once more) the time-independent Schrödinger equation. Explain for what physical property it is an eigenvalue equation. What is the meaning of the eigenvalues and eigenstates of this equation?

c) Again assume that the quantum system is a particle with mass m, that can only move in one direction (x-axis). It moves in a potential $V(x) = B_0 \cos(3x)$. Write down the time-independent Schrödinger equation for this case, using a representation where all states and operators are expressed as functions of x (that is, you must write it out in a form that shows each term of the equation).

Z.O.Z.

Problem T2
<u>For this problem, you must write up your answers in Dirac notation.</u>

Consider a quantum system that contains a charged particle with mass m, that has a time-independent Hamiltonian

$$\hat{H} = \hat{T} + \hat{V},$$

where T a kinetic-energy term and V a potential-energy term. With respect to a lowest point in the potential, defined as $V = 0$ J, the lowest two energy eigenstates of the system are defined by

$$\begin{aligned} \hat{H}|\varphi_1\rangle &= E_1|\varphi_1\rangle \\ \hat{H}|\varphi_2\rangle &= E_2|\varphi_2\rangle \end{aligned} ,$$

where $E_1 < E_2$ the two energy eigenvalues, and $|\varphi_1\rangle$ and $|\varphi_2\rangle$ two orthogonal, normalized energy eigenvectors. The observable $\hat{A}$, is associated with the magnetic moment A of this quantum system. For this system,

$$\langle\varphi_1|\hat{A}|\varphi_1\rangle = 3A_0 \quad , \quad \langle\varphi_2|\hat{A}|\varphi_2\rangle = -3A_0 \quad ,$$
$$\langle\varphi_n|\hat{A}|\varphi_m\rangle = \langle\varphi_m|\hat{A}|\varphi_n\rangle = A_0 \quad , \quad \text{for all cases } n \neq m.$$

Note that the states $|\varphi_1\rangle$ and $|\varphi_2\rangle$ are energy eigenvectors, and that they are ***not*** eigen vectors of $\hat{A}$.

a) What can you say about the possible values of E_1? Discuss the sign, whether it can be zero. Explain your answer.

b) At some time, the state of the system is (with all c_n a complex-valued constant)

$$|\Psi_S\rangle = c_1|\varphi_1\rangle + c_2|\varphi_2\rangle = \frac{1}{\sqrt{3}}|\varphi_1\rangle + \sqrt{\frac{2}{3}}\, e^{i\phi}\, |\varphi_2\rangle \quad ,$$

where ϕ the phase of the superposition state. Prove that this is a normalized state.

c) What is for this state $|\Psi_S\rangle$ the expectation value $\langle\hat{A}\rangle$ for A, expressed in A_0?

d) At some other time, defined as $t = 0$, the normalized state of the system is (with again all c_n a complex-valued constant)

$$|\Psi_0\rangle = c_1|\varphi_1\rangle + c_2|\varphi_2\rangle = -\frac{i\sqrt{15}}{4}|\varphi_1\rangle + \frac{i}{4}|\varphi_2\rangle \quad .$$

Show that as a function of time $t > 0$, the expectation value for $\langle\hat{A}\rangle$ has oscillations at one frequency only. Determine this frequency and the oscillation amplitude (expressed in the constants that are mentioned above), for the case that the system is in $|\Psi_0\rangle$ at $t = 0$. Use the time-evolution operator (with $\hbar = h/2\pi$)

$$\hat{U} = e^{\frac{-i\hat{H}t}{\hbar}} .$$

Problem T3
A wide parallel beam of electrons (only motion in y-direction) with a velocity of $v_y = 600$ m/s is incident on a screen with a single narrow slit of with d. Behind this first screen there is a second screen for detection. The distance between the screen with the slit and the detection screen is $l = 1$ m. Using the uncertainty principle, make an estimate for the width of the slit d for which the width W of the image on the detection screen is the narrowest. Hint: For electrons that just passed the screen, you can assume that for transverse motion in the beam, the state of electrons is close to a state with minimum uncertainty.

Antwoorden midtermtoets Kwantumfysica 1, 10 december 2010

Problem T1

a) Use for example x-representation, for case that the single degree of freedom is the position in x direction of particle with mass m (but you can work it out in a similar way for any other single degree of freedom)

Time-dependent Schrödinger equation: $i\hbar \frac{\partial}{\partial t} \Psi(x,t) = \hat{H}\Psi(x,t)$ (1)

with for this case $\boxed{\hat{H} = V(x) - \frac{\hbar^2}{2m}\frac{d^2}{dx^2}}$.

Investigate whether there are solutions of the type $\Psi(x,t) = \varphi(x)\chi(t)$

Filling this in into Eq. (1), and deviding by $\varphi(x)\chi(t)$ gives

$$\frac{i\hbar}{\chi(t)}\frac{d\chi(t)}{dt} = \frac{1}{\varphi(x)}\left(-\frac{\hbar^2}{2m}\frac{d^2\varphi(x)}{dx^2}\right) + V(x)$$

This equality can only hold (left function of t only, right function of x only) if left and right are equal to a constant, which will be denoted as E_i. This gives two equations

$$\begin{cases} \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right)\varphi(x) = \hat{H}\varphi(x) = E_i\varphi(x) \rightarrow \text{time-independent Schrödinger Eq.} \\ i\hbar\frac{d\chi(t)}{\chi(t)} = E_i\, dt \Rightarrow i\hbar \ln(\chi(t)) = E_i \cdot t + C \Rightarrow \chi(t) = e^{-\frac{i}{\hbar}(E_i t + C)} \rightarrow \end{cases}$$

time evolution of states with fixed E_i.

b) $\hat{H}\varphi(x) = E_i\varphi(x)$ $\Rightarrow$ Eigenvalue problem for operator $\hat{H}$ $\rightarrow$ associated with the system's total energy.

The eigen states (and values) represent physical states with a well-defined value for total energy (the associated energy eigenvalue) $\rightarrow$ These can be measurement outcomes if you measure the energy in the system.

c) $\hat{H}\varphi(x) = E_i\varphi(x) \Rightarrow -\frac{\hbar^2}{2m}\frac{d^2\varphi(x)}{dx^2} + B_0\cos(3x)\varphi(x) = E_i\varphi(x)$

②

Problem T2 a) For the groundstate $|\varphi_1\rangle$, the following must be valid:

$\langle\hat V\rangle = \langle\varphi_1|\hat V|\varphi_1\rangle > 0\,J$, since $V = 0\,J$ is lowest point in the potential

$\langle\hat T\rangle = \langle\varphi_1|\hat T|\varphi_1\rangle > 0\,J$, since kinetic energy is always $> 0\,J$

So, we must have $E_1 = \langle\varphi_1|\hat H|\varphi_1\rangle > 0\,J$.
The Heisenberg uncertainty relation forbids $E_1 = 0\,J$.

b) Need to show that $\langle\psi_s|\psi_s\rangle = 1$, and we can use that $\langle\varphi_1|\varphi_1\rangle = 1$, $\langle\varphi_2|\varphi_2\rangle = 1$, $\langle\varphi_2|\varphi_1\rangle = \langle\varphi_1|\varphi_2\rangle = 0$.

$$\langle\psi_s|\psi_s\rangle = \left(c_1^*\langle\varphi_1| + c_2^*\langle\varphi_2|\right)\left(c_1|\varphi_1\rangle + c_2|\varphi_2\rangle\right) = c_1^* c_1 + c_2^* c_2 =$$
$$\frac{1}{\sqrt3}\cdot\frac{1}{\sqrt3} + \sqrt{\tfrac{2}{3}}\,e^{-i\varphi}\sqrt{\tfrac{2}{3}}\,e^{i\varphi} = \frac13 + \frac23 = 1 .$$

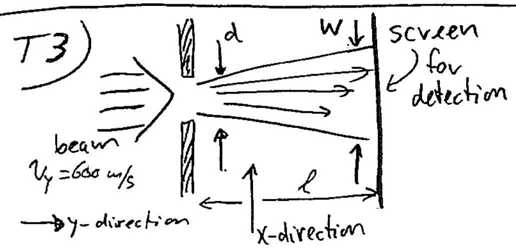
c) $$\langle\hat A\rangle = \langle\psi_s|\hat A|\psi_s\rangle = \left(c_1^*\langle\varphi_1| + c_2^*\langle\varphi_2|\right)\hat A\left(c_1|\varphi_1\rangle + c_2|\varphi_2\rangle\right) =$$
$$c_1^* c_1\langle\varphi_1|\hat A|\varphi_1\rangle + c_1^* c_2\langle\varphi_1|\hat A|\varphi_2\rangle + c_2^* c_1\langle\varphi_2|\hat A|\varphi_1\rangle + c_2^* c_2\langle\varphi_2|\hat A|\varphi_2\rangle$$
$$= \frac13\cdot 3A_0 + \left(\frac{1}{\sqrt3}\sqrt{\tfrac23}\,e^{i\varphi} + \frac{1}{\sqrt3}\sqrt{\tfrac23}\,e^{-i\varphi}\right)A_0 + \frac23\cdot(-3A_0)$$
$$= -A_0 + \frac{2\sqrt2}{3}\cos\varphi\cdot A_0 = \left(\frac{2\sqrt2}{3}\cos\varphi - 1\right)A_0$$

d) $\langle\hat A\rangle(t) = \langle\psi(t)|\hat A|\psi(t)\rangle$, with $|\psi(t)\rangle = \hat U|\psi_0\rangle = e^{-\frac{i}{\hbar}\hat H t}|\psi_0\rangle$
$\langle\psi(t)| = \langle\psi_0|\hat U^+ = \langle\psi_0|e^{+\frac{i}{\hbar}\hat H t}$

$$\langle\hat A\rangle(t) = \langle\psi_0|\hat U^+\hat A\hat U|\psi_0\rangle = \left(c_1^* e^{+iE_1 t/\hbar}\langle\varphi_1| + c_2^* e^{+iE_2 t/\hbar}\langle\varphi_2|\right)\hat A\left(c_1 e^{-iE_1 t/\hbar}|\varphi_1\rangle + c_2 e^{-iE_2 t/\hbar}|\varphi_2\rangle\right)$$
$$= |c_1|^2\langle\varphi_1|\hat A|\varphi_1\rangle + |c_2|^2\langle\varphi_2|\hat A|\varphi_2\rangle + \left(c_1^* c_2 e^{\frac{i(E_1-E_2)t}{\hbar}} + c_2^* c_1 e^{\frac{i(E_2-E_1)t}{\hbar}}\right)\langle\varphi_1|\hat A|\varphi_2\rangle$$
$$= \frac{15}{16}3A_0 - \frac{1}{16}3A_0 + \left(-\frac{1}{16}\sqrt{15}A_0\right)\left(2\cos\left(\frac{E_2-E_1}{\hbar}\cdot t\right)\right)$$
$$= \frac{21}{8}A_0 - \frac18\sqrt{15}\cos\left(\left(\frac{E_2-E_1}{\hbar}\right)t\right)A_0$$

Amplitude is $\frac18\sqrt{15}A_0$, Only frequency is $f = \frac{E_2-E_1}{2\pi\hbar}$, $(\omega = 2\pi f)$

T3)



Right after the screen with the slit $\Delta x \simeq d \Rightarrow$
$\Delta p_x \approx \frac{\hbar}{2\Delta x} = \frac{\hbar}{2d}$

While flying to the detection screen, the beam will get wider from **d** to a width $W = d + \Delta W$.

$\Delta W = \Delta v_x\cdot t = \frac{\Delta p_x}{m}\cdot t$, where t the time of flight between the two screens. $\Rightarrow t = \frac{\ell}{v_y} \Rightarrow$

$W = d + \Delta W = d + \frac{\hbar}{2md}\cdot\frac{\ell}{v_y} \Rightarrow W$ has a minimum for a certain d. To find this d solve $\frac{dW}{dd} = 0 \Rightarrow$

$$1 - \frac{1}{d^2}\frac{\hbar\ell}{2mv_y} = 0 \Rightarrow d \approx \sqrt{\frac{\hbar\ell}{2mv_y}}$$